Vocal Tract Models, Formant Frequencies, and Computational Methods

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0. Abstract

The determination of formant (or 'pole') and zero frequencies from area and length data is considered for two- and three-tube vocal tract models. A computer-implemented algorithm is shown to represent an improvement over graphical solution techniques because a discontinuity test is easily included.

1. Introduction

A reasonable approximation to the shape of the human vocal tract can be obtained by considering the tract to consist of a sequence of tubes. Each tube has uniform cross-sectional area $A$ over its entire length $\ell$. The model can then be made arbitrarily complex by increasing the number of such tubes to approximate more accurately the actual anatomy of the vocal tract. The model with the simplest geometry, a one-tube model, allows us to concentrate on the investigation of the various factors affecting the mode pattern in the vocal tract (e.g. the radiation load on the open end of the tube, glottal impedance, the vibration of the vocal tract walls, etc.). Its main problem is that it is a realistic approximation only for $/\omega/$. A full discussion of the one-tube model can be found in Flanagan (1972:58-69). Here we will be concerned with two-tube and three-tube models only. The same principles apply to more complex models, but the equations become correspondingly more complex, obscuring much of the basic methodology involved.

For any given vocal tract model, there are two basic types of numerical problem which can be examined:

1) The calculation of the formant frequencies from the model. Here the values of $A$ and of $\ell$ for each tube are known, and we proceed to calculate the desired members of the countably infinite set (Grever 1967:8) of formant frequencies associated with that particular geometrical configuration. This set of formants ('spectral peaks in the output sound' Flanagan 1972:59) can be represented symbolically as $F = \{F_1, F_2, F_3, \ldots\}$. In this first problem, there
is always a unique solution for each $F_1$, although the actual process of calculating this solution may involve difficult computational problems.

2) The calculation of the $A$ and $l$ parameters of the model from a set of known formant frequencies. In this second problem, there is no unique solution for each value of $A$ and each value of $l$; we can only determine the ratio in which the $A$ values must stand relative to one another (Mermelstein 1967:1287) and the ratio in which the $l$ values must stand relative to one another. Of course, we can then determine unique solutions for the values of $l$ by fixing their sum as the total vocal tract length. However, we can never determine more than an $A$ ratio, and hence the set of $A$ values remains uncountably infinite (Greever 1967:8).

Both of these problems will now be examined with respect to the two-tube and three-tube models presented in Flanagan (1972:69-79).

2. Two-tube Model

Flanagan makes the simplifying assumptions that the tubes are lossless, that the glottal impedance is high compared with the input impedance of the tract, and that the radiation load is negligible compared with the impedance level at the mouth (i.e. glottal impedance is assumed to be infinite and radiation impedance to be zero). The model can be represented as

![Two-tube Model](Figure 1)

where the subscript 1 refers to the back cavity and the subscript 2 to the front cavity. The formants ($\beta$) satisfy

(1) \[ \frac{A_1}{A_2} \tan\beta_2 l_2 = \cot\beta_1 l_1 \] \hspace{1cm} \text{(Flanagan 1972:70)}

Note that we could write

(2) \[ \frac{A_1}{A_2} \tan\beta_2 \tan\beta_1 l_2 = 1 \]

which makes more explicit the fact that provided the area ratio of
the two cavities is maintained constant, their lengths can be interchanged without altering the formant frequencies (i.e. the values of $\beta$ which satisfy Equation 2 are the same for $l_1=a$, $l_2=b$ as for $l_1=b$, $l_2=a$).

Equation 1 holds for excitation at any point in the system; in particular, it holds for the glottal excitation in vowels as well as for the forward excitation in fricatives. Zeros do not occur for glottal excitation, but only for forward excitation: these zeros satisfy

$$3 \tan \beta l_1 = -A_2 \tan \beta l_2 \frac{A_2}{A_1}$$

Let us now turn to the two numerical problems mentioned in Section 1 above. The calculations are relatively complex, and become more so if some of the simplifying assumptions are relaxed or if a third tube is added; hence the logical means of solution is to use a computer. Flanagan 1972 uses the traditional graphical techniques, the primary and widespread means for solving such equations in mathematics before the advent of computers. Such techniques are less accurate than computer solution, although obviously there is little point in using the full power of computerized numerical methods to calculate formant frequencies to many decimal places from a mere two- or three-tube model of the vocal tract. A more serious problem with the graphical technique is that errors may result from failure to discern points of discontinuity in the tangent and cotangent functions: the possibility of such errors is eliminated by including a discontinuity test in the computer program. In addition, a computer program is more versatile because it allows one to change the configuration of the vocal tract (within the limits of the two-tube model) and determine the new solutions effortlessly, whereas the graphical technique requires a completely new graph to be constructed for each new configuration.

The first problem is to determine the set $F = \{F_1, F_2, F_3, \ldots\}$, given values for $A_1, A_2, l_1, l_2$. The zeros, $Z = \{Z_1, Z_2, Z_3, \ldots\}$ may also be calculated when the excitation is forward in the vocal tract (i.e. for fricatives). A brief description of some aspects of
the program follows; a flow-chart may be found in Appendix 2. Only
the first five zeros and the first five poles have been computed;
fewer or more than five can of course be computed by altering the
limiting statement at the end of each iteration section.

Input to the program is the set of numerical values \( A_1, A_2, l_1, \)
and \( l_2 \), followed by any non-zero digit if there is forward
excitation in the vocal tract. If no non-zero digit is present,
this indicates glottal excitation, and the section of the program
computing the zeros of transmission is accordingly bypassed (i.e.
this digit is simply acting as a 'flag'). The poles are computed
first, using an iterative approximation technique applied to Equation
1, which can be rewritten as

\[
(4) \quad A_1 \tan \beta l_2 = A_2 \cot \beta l_1 = 0
\]

By increasing \( \beta \) gradually from \( 0^+ \) (i.e. from a positive number very
close to 0), we obtain a sequence of values for the left-hand
side of Equation 4. This sequence is the set of values of the
variable called TEST in the program. If two successive values
of TEST differ in sign, we know that, for some intermediate value
of \( \beta \), the value of TEST must have been zero, provided that the
change in sign did not result from a discontinuity. This is a
particularly important point, and the ease with which we can
check whether the change in sign is merely due to a discontinuity
is one of the principal advantages of the computer solution over
the graphical solution. For any \( \theta \), if \( \tan \theta \) is positive but
\( \tan(\theta+\delta\theta) \) (where \( \delta\theta \) is positive, but very small) is negative, then
the tangent function has a discontinuity in the interval \((\theta,\theta+\delta\theta)\).
Similarly, if \( \cot\theta \) is negative but \( \cot(\theta+\delta\theta) \) is positive, then the
cotangent function has a discontinuity in the interval \((\theta,\theta+\delta\theta)\).

In either case we simply continue our iteration. If in fact the
value of TEST has truly been zero in the interval we can interpolate
a more exact value of \( \beta \), and then return to the iterative procedure
to compute the next value of \( \beta \) for which TEST is zero. For a fuller
explanation of tangent and cotangent discontinuities, see Appendix 3.

When the desired number of poles has been computed, the
program then turns to the computation of the zeros (provided, of
course, that forward excitation has been indicated), by an identical
procedure applied to Equation 3, which can be rewritten as

\[ A_1 \tan \beta \lambda_1 + A_2 \tan \beta \lambda_2 = 0 \]

Note that the poles and zeros have been calculated in terms of \( \beta \). These must subsequently be converted to frequency values, using the relationship

\[ f = \frac{\beta c}{2\pi} \]

which can be derived from Flanagan (1972:79) \((c \text{ is the speed of sound in the vocal tract in cm/second; } f \text{ is the frequency in Hz})\). This program performs the conversion immediately following the calculation of each pole or zero, overwriting the frequency value in place of the \( \beta \) value.

Another feature of this program is that the size of the increment in \( \beta \) between each iteration is variable. As long as we are relatively distant from a value of \( \beta \) for which TEST equals zero, we can allow the \( \beta \)-increment to be larger than when we are closer to a value where TEST equals zero. This will save a considerable amount of execution time (the main difficulty with iterative techniques is that they are often time-consuming), while preserving the accuracy of our determination of \( \beta \).

Incidentally, how accurate is our determination of \( \beta \)? The smallest \( \beta \)-increment is 0.0001 and we then interpolate. Thus we are at least as accurate as 0.00005 or

\[ \frac{(35000) (0.00005)}{2\pi} \approx 0.28 \text{ Hz} \]

(from Equation 6)

The program was tested using data from Flanagan (1972:71,74). The following results were obtained:

\[
/1/ \quad A_1=8\text{cm}^2 \quad A_2=1\text{cm}^2 \quad \lambda_1=9\text{cm} \quad \lambda_2=6\text{cm}
\]

Poles: Flanagan's values: 250 1875 2825 (Hz)
My values: 256.8 1904.1 2916.7 3938.0 5576.6 (Hz)

Zeros: Flanagan's values: glottal excitation and hence no zeros
My values: 

\( /\alpha/ \quad A_1 = 1 \text{cm}^2 \quad A_2 = 8 \text{cm}^2 \quad l_1 = 4 \text{cm} \quad l_2 = 13 \text{cm} \)

### Poles: Flanagan's values:

- 625
- 1700
- 2325

### My values:

- 646.3
- 1829.2
- 2358.5
- 4690.1
- 5947.0

### Zeros: Flanagan's values:

- glottal excitation and hence no zeros

### My values:

- glottal excitation and hence no zeros

\( /a/ \quad A_1 = 1 \text{cm}^2 \quad A_2 = 7 \text{cm}^2 \quad l_1 = 9 \text{cm} \quad l_2 = 8 \text{cm} \)

### Poles: Flanagan's values:

- 750
- 1250
- 2700

### My values:

- 788.9
- 1275.8
- 2808.3
- 3385.6
- 4799.6

### Zeros: Flanagan's values:

- glottal excitation and hence no zeros

### My values:

- glottal excitation and hence no zeros

\( /\varepsilon/ \quad A_1 = 6 \text{cm}^2 \quad A_2 = 6 \text{cm}^2 \quad l_1 = 17 \text{cm} \quad l_2 = 0 \text{cm} \)

### Poles: Flanagan's values:

- 500
- 1500
- 2500

### My values:

- 514.7
- 1544.1
- 2573.5
- 3602.9
- 4632.4

### Zeros: Flanagan's values:

- glottal excitation and hence no zeros

### My values:

- glottal excitation and hence no zeros

\( /s/ \quad A_1 = 7 \text{cm}^2 \quad A_2 = 0.2 \text{cm}^2 \quad l_1 = 12.5 \text{cm} \quad l_2 = 2.5 \text{cm} \)

### Poles: Flanagan's values:

- 160
- 1375
- 2725
- 4080
- 5440
- 6650
- 6950

### My values:

- 164.4
- 5582.8
- 6835.6
- 7164.4
- 12582.5

### Zeros: Flanagan's values:

- 1350
- 2675
- 3400
- 4100
- 5440
- 6800

### My values:

- 1390.8
- 2763.0
- 4237.0
- 5609.2
- 7000.0

Agreement between Flanagan's values and my values for /i/, /\alpha/,
/a/, and /\varepsilon/ is in most cases very close. In cases where the agreement
is not quite so close, my values give a better agreement between
the left- and right-hand sides when substituted in Equation 1. However,
for /s/, Flanagan's \( F_2 \), \( F_3 \), and \( F_4 \) as well as \( Z_3 \) do not correspond to
any of my poles or zeros. These values found by Flanagan all involve
discontinuities in the tangent or cotangent functions. This can be
seen either by a careful examination of Flanagan's graphs, or by
substituting Flanagan's values into the appropriate equation (Equation
4 or 5).

<table>
<thead>
<tr>
<th>( f ) (Hz)</th>
<th>( \frac{\beta}{2\pi f} )</th>
<th>Equation</th>
<th>Left-Hand Side</th>
<th>Right-Hand Side</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F_2 )=1375</td>
<td>0.2468</td>
<td>4</td>
<td>8.4960</td>
<td>0</td>
</tr>
<tr>
<td>( F_3 )=2725</td>
<td>0.4892</td>
<td>4</td>
<td>20.4865</td>
<td>0</td>
</tr>
<tr>
<td>( F_4 )=4080</td>
<td>0.7324</td>
<td>4</td>
<td>-25.5688</td>
<td>0</td>
</tr>
<tr>
<td>( Z_3 )=3400</td>
<td>0.6104</td>
<td>5</td>
<td>35.1898</td>
<td>0</td>
</tr>
</tbody>
</table>
Thus the superior ability of the computer program (over the graphical technique) to discern apparent solutions due to discontinuities from genuine solutions has been demonstrated.

Remaining with a two-tube model, we can now examine the other type of problem mentioned above, namely the estimation of $A_1$, $A_2$, $l_1$, and $l_2$ given some of the formant frequencies. Using the following method, just two formant frequencies are sufficient; any two formant frequencies can be used, without specifying which formants they represent. We have two given formant frequencies, which can be converted to $\beta$-values using Equation 6. Thus we have two values, $\beta_1$ and $\beta_2$, for which Equation 1 holds:

\begin{align}
(7a) \quad A_1 \tan \beta_1 l_2 &= \cot \beta_1 \frac{l_1}{A_2} \\
(7b) \quad A_1 \tan \beta_2 l_2 &= \cot \beta_2 \frac{l_1}{A_2}
\end{align}

Dividing Equation 7a by Equation 7b, we have

\begin{align}
(8) \quad \frac{\tan \beta_1 l_2}{\tan \beta_2 l_2} &= \frac{\cot \beta_1 l_1}{\cot \beta_2 l_1}
\end{align}

If we also assume that the total length of the vocal tract is 17 cm, we have

\begin{align}
(9) \quad l_1 + l_2 &= 17
\end{align}

Combining Equations 8 and 9, we have

\begin{align}
(10) \quad \frac{\tan \beta_1 (17 - l_1)}{\tan \beta_2 (17 - l_1)} - \frac{\cot \beta_1 l_1}{\cot \beta_2 l_1} &= 0
\end{align}

which, despite its formidable appearance, has only one unknown ($l_1$).

This can now be solved on the computer by an iterative procedure for a value of $l_1$; $l_2$ is then obtained by substitution in Equation 9. The $A$-ratio, $A_1/A_2$, can then be obtained from either Equation 7a or Equation 7b; Equation 7a was used here.

There are two major points to be noted in connection with this program. The first is that for our purposes Equation 1 must be restricted to allow only a positive value of $A_1/A_2$, since physically a negative $A$-ratio would be meaningless. Mathematically no such
restriction exists; thus our solution for \( l_1 \) and \( l_2 \) may produce a negative \( A \)-ratio. If this happens, we must regard this solution \( l_1 \) and \( l_2 \) as 'inadmissible', and continue with our iteration until we find an admissible solution. The second point is that there is no need to vary the value of the \( l_1 \)-increment in our iteration procedure to obtain a more accurate value of \( l_1 \), since to the nearest 0.05 cm is already more accurate than necessary.

The program was tested using data from Flanagan (1972:71). Some differences result from the fact that Flanagan's data are not constrained to have \( l_1 + l_2 = 17 \) cm. This program is really for demonstration purposes only; a more practical program would probably take several formants as input, use these formants pairwise to compute values of \( l_1, l_2 \), and the \( A \)-ratio, and then use an averaging procedure to give a combined approximate solution.

3. Three-tube Model
Let us now turn to the three-tube mode. This allows us to model nasal consonants and nasalized vowels in addition. The model can be represented as

\[ \begin{align*}
A_m & \quad \text{Figure 2} \\
A_p & \\
\ell_p & \\
\ell_m & \\
\end{align*} \]

where the subscript \( m \) refers to the mouth cavity, \( n \) to the nasal cavity, and \( p \) to the pharynx. The formants satisfy

\[
(11) \quad A_m \tan \beta \ell_m + A_p \tan \beta \ell_p - A \cot \beta \ell_n = 0
\]

and the zeros satisfy

\[
(12) \quad f = (2n + 1) \frac{c}{4 \ell_m}
\]

when the system is lossless (Flanagan 1972:78, 79).
The first numerical problem is to calculate pole and zero frequencies, given the values of \( A_m \), \( A_n \), \( A_p \), \( l_m \), \( l_n \), and \( l_p \). Again, only the first five zeros and poles have been computed, but this number can easily be changed. The calculation of the zeros is a simple matter, in fact, far simpler than for the two-tube model. The equation for the poles is more complicated, and would require considerably more effort to solve graphically than the corresponding two-tube model equation (i.e. Equation 1). However, we can use the same basic algorithm to solve the problem numerically by computer, namely iterative approximation on \( \beta \). As before, we allow the size of the \( \beta \)-increment to vary depending upon our distance from the solution, and perform an interpolation on the final \( \beta \) values to improve the accuracy.

A flowchart is given in Appendix 2. The following results were obtained, using area and length data from Flanagan (1972:78):

<table>
<thead>
<tr>
<th>/m/</th>
<th>( A_m = 6 cm^2 )</th>
<th>( A_n = 4 cm^2 )</th>
<th>( A_p = 5 cm^2 )</th>
<th>( l_m = 6.5 cm )</th>
<th>( l_n = 12.5 cm )</th>
<th>( l_p = 8.5 cm )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Poles: Flanagan's values:</td>
<td>250</td>
<td>1100</td>
<td>1350</td>
<td>2200</td>
<td>(Hz)</td>
<td></td>
</tr>
<tr>
<td>My values:</td>
<td>309</td>
<td>1137.3</td>
<td>2268.1</td>
<td>2896.9</td>
<td>3473.4</td>
<td>(Hz)</td>
</tr>
<tr>
<td>Zeros: Flanagan's values:</td>
<td>1300</td>
<td>3900</td>
<td>6500</td>
<td>9100</td>
<td>11700</td>
<td>(Hz)</td>
</tr>
<tr>
<td>My values:</td>
<td>1346.2</td>
<td>4038.5</td>
<td>6730.8</td>
<td>9423.1</td>
<td>12115.4</td>
<td>(Hz)</td>
</tr>
</tbody>
</table>

Agreement between Flanagan's values and my values for the zeros and poles is reasonably good, with one striking exception. Flanagan finds a pole at 1350 Hz, whereas my program finds no corresponding pole. Further investigation indicates that this pole does not in fact exist. At \( f = 1350 \) Hz, \( \beta = 0.24235 \) and the left-hand side of Equation 11 is approximately \(-1311\), whereas the right-hand side is 0. In actual fact \( f = 1350 \) Hz represents a change in sign caused by a discontinuity, which Flanagan with his graphical technique has mistaken for a zero. This clearly points out the superiority of the computer solution. Another piece of evidence for the non-existence of a pole at 1350 Hz is the observed spectrum for \( /m/ \) which Flanagan (1972:80) gives without comment. The zeros and all the other poles can be recognized clearly in the spectrum, but there is nothing resembling
a pole near 1350 Hz. The observed spectrum then also reflects all the results obtained from the computer program.

The second numerical problem involving the three-tube model is the determination of $A_m/A_n$, $A_p/A_n$, $\ell_m/\ell_n$, and $\ell_p/\ell_n$ from observed formant frequencies. Thus, as before, we have several values of $\beta$ for which Equation 11 holds:

\begin{align*}
(13a) & \quad A_m \tan \beta_1 \ell_m + A_p \tan \beta_1 \ell_p - A_n \cot \beta_1 \ell_n = 0 \\
(13b) & \quad A_m \tan \beta_2 \ell_m + A_p \tan \beta_2 \ell_p - A_n \cot \beta_2 \ell_n = 0 \\
(13c) & \quad A_m \tan \beta_3 \ell_m + A_p \tan \beta_3 \ell_p - A_n \cot \beta_3 \ell_n = 0
\end{align*}

We can only determine area ratios; let us rewrite these equations to make this more explicit:

\begin{align*}
(14a) & \quad A_m \frac{\tan \beta_1 \ell_m}{A_n} + A_p \frac{\tan \beta_1 \ell_p}{A_n} - \cot \beta_1 \ell_n = 0 \\
(14b) & \quad A_m \frac{\tan \beta_2 \ell_m}{A_n} + A_p \frac{\tan \beta_2 \ell_p}{A_n} - \cot \beta_2 \ell_n = 0 \\
(14c) & \quad A_m \frac{\tan \beta_3 \ell_m}{A_n} + A_p \frac{\tan \beta_3 \ell_p}{A_n} - \cot \beta_3 \ell_n = 0
\end{align*}

As in the two-tube model, we can fix the total length of the vocal tract:

\[ \ell_m + \ell_p = 17 \]

Hence,

\begin{align*}
(16a) & \quad A_m \frac{\tan \beta_1 (17-\ell_p)}{A_n} + A_p \frac{\tan \beta_1 \ell_p}{A_n} - \cot \beta_1 \ell_n = 0 \\
(16b) & \quad A_m \frac{\tan \beta_2 (17-\ell_p)}{A_n} + A_p \frac{\tan \beta_2 \ell_p}{A_n} - \cot \beta_2 \ell_n = 0 \\
(16c) & \quad A_m \frac{\tan \beta_3 (17-\ell_p)}{A_n} + A_p \frac{\tan \beta_3 \ell_p}{A_n} - \cot \beta_3 \ell_n = 0
\end{align*}

This is still a rather complex system to solve, but if we also fix $\ell_n$ (e.g., at Flanagan's 12.5cm), we can reduce the number of unknowns to three (i.e., $\ell_p$, $A_p/A_n$, $A_m/A_n$). Iteration on $\ell_p$ would be required for the solution. For each successive value of $\ell_p$ we would check first to see if the three equations (Equations 16) were consistent (e.g., by computation of the determinant; see an elementary
text on matrix algebra for details). If they were consistent, then that value of \( l_p \) would represent a solution. We could then use any pair of equations from Equations 16 and Kramer's Rule (see a matrix algebra text) to solve for the area ratios \( A_m / A_n \) and \( A_p / A_n \). As with the two-tube model, we must observe the physical constraint that areas be positive; thus solutions with a negative area ratio must be rejected and iteration carried further.

Another approach can be used if we know the frequency value of one of the zeros and which zero it is. This allows us to use Equation 12 to calculate \( l_m \); Equation 15 can then be used to calculate \( l_p \). If \( l_n \) is also fixed, this reduces the complexity of the equations considerably, and a solution by matrix inversion is then easily carried out.

4. Conclusion

Three conclusions can be drawn from the above limited discussion of numerical problems involving tube models of the vocal tract. First, it is generally more straightforward to determine the pole and zero frequencies given the areas and lengths than vice versa. Second, as expected, the problems involved become more complex as the number of tubes in the model increases (Jospa 1974a, 1974b, 1975; Mermelstein 1967). Third, without the use of a computer and numerical methods techniques, the problems involved quickly become too complicated and arduous to solve by hand with any degree of accuracy and confidence.
Appendix 1

Throughout, the vocal tract has been assumed to have a total length of 17 cm (Flanagan 1972:60). This is appropriate to an adult male. Women and children of course have shorter tracts, and hence higher formant frequencies. If the female vocal tract is 0.87 as long as the male tract (Flanagan 1972:71), the female formants should be 1/0.87 or approximately 1.15 times those of the male.

I have also taken $c$, the speed of sound in the vocal tract, to be 35,000 cm/second (Flanagan 1972:69). My own calculations indicate a slightly higher figure (see below), but this does not much affect the dependent calculations. Assume the vocal tract is at the normal body temperature of $37^\circ$C. From Smith and Cooper 1964:209,

$$V_t = V_0 \sqrt{\frac{T}{273}}$$

where $T$ is absolute temperature (i.e. measured in degrees Kelvin) and $V_0 = 331.4$ m/second.

Here, $t=37$ and $T=273+37=310$; hence $V_t = 353.14$ m/second = 35314 cm/second. Or, from Beranek 1971:6,

$$c = 20.05 \sqrt{T} \text{ m/sec} = 35302 \text{ cm/sec},$$

where $T$ is absolute temperature.

Thus, the speed of sound in the vocal tract would be slightly over 35000 cm/second.
Appendix 2

This appendix includes a flow-chart for each computer program described in this paper. Copies of the programs and sample output for each program may be obtained from the author on request.

1. Calculation of formants and zeros for a two-tube model

```
START
READ INPUT DATA
(A1, A2, l1, l2, forward excitation)
PRINT INPUT DATA
INITIALIZATIONS
INCREMENT β, COMPUTE TEST FOR POLES
TEST HAS CHANGED SIGN IN THE INTERVAL
yes
DISCONTINUITY IN THE INTERVAL
yes
INTERPOLATE VALUE OF POLE
no
FIVE POLES HAVE BEEN COMPUTED
yes
PRINT VALUES OF POLES
no
ADJUST MAGNITUDE OF β INCREMENT

THERE IS FORWARD EXCITATION
yes
INITIALIZATIONS
INCREMENT β, COMPUTE TEST FOR ZEROS
TEST HAS CHANGED SIGN IN THE INTERVAL
yes
DISCONTINUITY IN INTERVAL
yes
INTERPOLATE VALUE OF ZERO
no
FIVE ZEROS HAVE BEEN COMPUTED
yes
PRINT VALUES OF ZEROS
no
ADJUST MAGNITUDE OF β INCREMENT

PRINT "NO ZEROS"

END
```
2. Calculation of area ratio and lengths for a two-tube model

START
READ INPUT DATA
(any two formant frequencies)
PRINT INPUT DATA
INITIALIZATIONS
CONVERT POLES IN
HZ TO POLES IN
TERMS OF β
INCREMENT \( \ell_1 \),
COMPUTE TEST
TEST
HAS CHANGED
SIGN IN THE
INTERVAL

yes
no

yes
no

DISCONTINUITY
IN THE INTERVAL

INTERPOLATE
VALUE OF \( \ell_1 \)
COMPUTE \( \ell_2 \)
COMPUTE A\(^2\)RATIO

A-RATIO
IS NEGATIVE

no
yes

PRINT \( \ell_1, \ell_2 \)
AND A-RATIO

END OF
DATA SETS

yes
no

END
3. Calculation of formant and zero frequencies for a three-tube model

START
READ INPUT DATA
\((A_m^A_n^A_p^l_m^l_n^l_p)\)
PRINT INPUT DATA
INITIALIZATIONS
COMPUTE FIRST FIVE ZEROS
PRINT FIRST FIVE ZEROS
INCREMENT \(\beta\), COMPUTE TEST FOR POLES

TEST HAS CHANGED SIGN IN THE INTERVAL
yes

DISCONTINUITY IN THE INTERVAL
yes
INTERPOLATE VALUE OF POLE

no

ADJUST MAGNITUDE OF \(\beta\) INCREMENT
FIVE POLES HAVE BEEN COMPUTED
yes

PRINT VALUES OF FIVE POLES

no

END
Appendix 3: Tangent and Cotangent Discontinuities

Tangent
(discontinuities at $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, ...$)

Cotangent
(discontinuities at $0, \pi, 2\pi, ...$)
References
Belmont, California: Brooks Cole.